On gluing a surface of genus g from one and two bicolored polygons

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The general task

We have k polygons $D_1, D_2, \ldots, D_k$. Vertices are properly colored black and white. On each polygon D_i we fix a counterclockwise orientation and mark one edge with the number i. Then we divide all edges into pairs and glue together correspondent edges with preserve the orientation and colors of the vertices.
What is the number of ways to glue a surface of genus g?

Definitions

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Let $B_g(m_1, \ldots, m_k; k)$ be the number of ways to glue together k bicolored polygons with $2m_1, \ldots, 2m_k$ edges into a connected orientable surface of genus g.

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Definition

Let $\mathcal{B}_g^{[k]}(z) = \sum\limits_{n\geq 0} B_g(n, k) z^n$ be the generation function of this numbers.

The case of one polygon

Harer-Zagier formula (J. Harer, D. Zagier, 1986)

Let $\varepsilon_g(n)$ be the number of ways to glue a surface of genus g from one $2n$-gon. Then

$$\varepsilon_g(n) = \frac{2n-1}{n+1}(2\varepsilon_g(n-1) + (n-1)(2n-3)\varepsilon_{g-1}(n-2)).$$

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Adrianov-Jackson formula (N. M. Adrianov, D. M. Jackson)

Let $B_g(n) = B_g(n, 1)$ be the number of ways to glue a surface of genus g from one bicolored $2n$-gon. Then

$$B_g(n) = \frac{1}{n+1}(2(2n-1)B_g(n-1) + (n-2)(n-1)^2 B_{g-1}(n-2)).$$

Generation functions in the case of one polygon

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Theorem (J. E. Andersen, R. C. Penner at al., 2010)

For $g > 0$ we have

$$\mathbf{C}_g(z) = \frac{P_g(z)}{(1-4z)^{3g-\frac{1}{2}}},$$

where $P_g(z)$ is a polynomial with integer coefficients of degree at most $3g-1$, divisible by z^{2g}. In addition, $P_g(\frac{1}{4}) \neq 0$.

The case of two polygons

Let $\varepsilon_g(n,k)$ be the number of ways to glue together k bicolored polygons with $2n$ edges in common into a connected orientable surface of genus g. Let $\mathbf{C}_g^{[k]}(z) = \sum_{n\geq 0} \varepsilon_g(n,k) z^n$ be the correspondent generation function.

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Theorem (J. E. Andersen, R. C. Penner at al., 2010)

For $g \geq 0$ we have

$$\mathbf{C}_g^{[2]}(z) = \frac{P_g^{[2]}(z)}{(1-4z)^{3g+2}},$$

where $P_g^{[2]}(z)$ is a polynomial with integer coefficients, which satisfies the equation

$$P_g^{[2]}(z) = z^{-1} P_{g+1}(z) - \sum_{h=1}^{g} P_h(z) P_{g+1-h}(z).$$

The polynomial $P_g^{[2]}(z)$ has degree at most $3g+1$ and divisible by z^{2g+1}. In addition, $P_g^{[2]}(\frac{1}{4}) > 0$.

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Some explicit formulas

$$\varepsilon_0(n,2) = n4^{n-1}; \quad \varepsilon_1(n,2) = \frac{1}{12}(13n+3)n(n-1)(n-2)4^{n-3}.$$

These two formulas could be derived from the formulas for $\mathbf{C}_g^{[2]}(z)$. Another proof was found by A.P. and O. P. Rodionova, 2012.

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It would be interesting to find a direct bijection between these gluings!

Much more general task, connected with bicolored gluings, was investigated in a preprint of P. Zograf (arXiv:1312.2538).

The main result

Theorem

For $g > 0$ we have

$$\mathcal{B}_g(z) = \frac{Q_g(z)}{(1-4z)^{3g-\frac{1}{2}}},$$

where $Q_g(z)$ is a polynomial with integer coefficients of degree $4g-1$, divisible by z^{2g+1}, and $Q_g(\frac{1}{4}) > 0$.

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The first theorem

From the Adrianov-Jackson formula

$$B_g(n) = \frac{1}{n+1}(2(2n-1)B_g(n-1) + (n-2)(n-1)^2 B_{g-1}(n-2)),$$

it follows a differential equation

$$(1-2z)\mathcal{B}_g(z) + (z-4z^2)\mathcal{B}'_g(z) = z^3(\mathcal{B}'_{g-1}(z) + 5z\mathcal{B}''_{g-1}(z) + z^2\mathcal{B}'''_{g-1}(z)).$$

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Then we have

$$\mathcal{B}_g(z) = \frac{2z^2(2z^2-z+2)}{(1-4z)^3}\mathcal{B}_{g-1}(z) - \frac{6z^4}{(1-4z)^2}\mathcal{B}'_{g-1}(z) +$$
$$+ \frac{z^4}{1-4z}\mathcal{B}''_{g-1}(z) + \frac{(1-4z)^{\frac{1}{2}}}{z} \cdot \int_0^z \frac{12t^2(2t^3-2t^2-1)}{(1-4t)^{\frac{9}{2}}}\mathcal{B}_{g-1}(t)\,dt.$$

The first theorem

It's well known, that $\mathcal{B}_0(z) = \frac{1-\sqrt{1-4z}}{2z}$. Then we have

$$\mathcal{B}_1(z) = \frac{z^3}{(1-4z)^{2,5}},$$

and by induction

$$\mathcal{B}_g(z) = \frac{Q_g(z)}{(1-4z)^{3g-\frac{1}{2}}},$$

where

$$\begin{aligned} Q_g(z) = 2z^2(6(12g^2 - 30g + 19)z^2 - z + 2)Q_{g-1}(z) + \\ + 2(12g-17)z^4(1-4z)Q'_{g-1}(z) + z^4(1-4z)^2 Q''_{g-1}(z) + \\ + \frac{(1-4z)^{3g}}{z} \int\limits_0^z \frac{12t^2(2t^3 - 2t^2 - 1)}{(1-4t)^{3g+1}} Q_{g-1}(t)\, dt. \end{aligned}$$

The first theorem

- $Q_1(z) = z^3$;
- $Q_2(z) = 9z^7 - 8z^6 + 8z^5$;
- $Q_3(z) = 450z^{11} - 720z^{10} + 528z^9 - 32z^8 + 180z^7$;
- $Q_4(z) = 55125z^{15} - 126000z^{14} + 130032z^{13} - 77312z^{12} + 42744z^{11} + 15552z^{10} + 8064z^9$;
- $Q_5(z) = 12502350z^{19} - 37044000z^{18} + 50349600z^{17} - 41302080z^{16} + 22847896z^{15} - 7281088z^{14} + 7649856z^{13} + 3035520z^{12} + 604800z^{11}$.

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Coefficients at z^{4g-1} and z^{2g+1}.

$$a(g) = C^{g}_{2g-1}(2g-1)!!(2g-3)!!; \quad b(g) = \frac{(2g)!}{g+1}.$$

$$Q_g\left(\frac{1}{4}\right) = \frac{2(6g-3)!!}{384^g g!}.$$

The second theorem

The operation of deleting an edge

Let $\mathcal{M} = (X, G)$ be a connected bicolored marked map of genus g with v vertices, $n+1$ edges and k faces, where $n > 0$. Let e_1 be a marked arc of the first face, f_1 be the arc opposed to e_1 and $\tilde{e}_1$ be the correspondent edge of G. Let's delete the edge $\tilde{e}_1$ from the graph G. Then we have several cases.

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Case 1: e_1 and f_1 belongs to different faces

We get a connected bicolored map of genus g with v vertices, n edges and $k - 1$ faces. We need to change some marks to get a marked map. Then any such marked map, in which the face 1 has $2m$ arcs, is obtained $\frac{m(m+1)(k-1)}{2}$ times.

The second theorem

Case 2: e_1 and f_1 belongs to a same face and graph $G - \tilde{e}_1$ is connected

We get a connected bicolored map of genus $g - 1$ with v vertices, n edges and $k + 1$ faces. We change some marks and find that any such marked map is obtained once.

The second theorem

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We get a connected bicolored map of genus $g - 1$ with v vertices, n edges and $k + 1$ faces. We change some marks and find that any such marked map is obtained once.

Case 3: e_1 and f_1 belongs to a same face and graph $G - \tilde{e}_1$ is disconnected

We get a pair of connected bicolored marked maps of genus $g_1 + g_2 = g$ with v vertices, n edges and $k + 1$ faces in common. We change some marks and find that any such pair of marked maps with k_1 and k_2 faces is obtained $C_{k-1}^{k_1-1} = C_{k-1}^{k_2-1}$ times.

The second theorem

The main recurrence

For any integers $n > 0$, $k > 0$ and $g \geq 0$ the following equality holds:

$$B_g(n+1,k) = \sum_{m_1+\ldots+m_{k-1}=n} \tfrac{m_1(m_1+1)(k-1)}{2} B_g(m_1,\ldots,m_{k-1};k-1) +$$

$$+ \sum_{\ell=1}^{k} \sum_{h=0}^{g} \sum_{i=0}^{n} C_{k-1}^{\ell-1} B_h(i,\ell) B_{g-h}(n-i,k-\ell+1) +$$

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$$+\sum_{\ell=1}^{k}\sum_{h=0}^{g}\sum_{i=0}^{n} C_{k-1}^{\ell-1} B_h(i,\ell) B_{g-h}(n-i,k-\ell+1)+$$

$$+B_{g-1}(n,k+1).$$

Corollary

$$B_g(n,2) = B_{g+1}(n+1) - \sum_{h=0}^{g+1}\sum_{i=0}^{n} B_h(i) B_{g+1-h}(n-i).$$

The second theorem

From the corollary and the formula for $\mathcal{B}_g(z)$ it's easy to see that

$$\mathcal{B}_g^{[2]}(z) = \frac{Q_g^{[2]}(z)}{(1-4z)^{3g-\frac{1}{2}}},$$

where the polynomial $Q_g^{[2]}(z)$ can be calculated by formula

$$Q_g^{[2]}(z) = z^{-1}Q_{g+1}(z) - \sum_{h=1}^{g} Q_h(z)Q_{g+1-h}(z).$$

$Q_g^{[2]}(z)$ is a polynomial with integer coefficients of degree at most $4g+2$, divisible by z^{2g+2}, and $Q_g^{[2]}(\frac{1}{4}) > 0$.

The second theorem

- $Q_0^{[2]}(z) = z^2$;
- $Q_1^{[2]}(z) = 8z^7 - 8z^6 + 8z^5$;
- $Q_2^{[2]}(z) = 432z^{10} - 704z^9 + 512z^8 - 32z^7 + 180z^6$;
- $Q_3^{[2]}(z) = 54144z^{14} - 124416z^{13} + 128768z^{12} - 77120z^{11} + 42320z^{10} + 15552z^9 + 8064z^8$;
- $Q_4^{[2]}(z) = 2384000z^{18} - 36771840z^{17} + 50061312z^{16} - 41126912z^{15} + 22750208z^{14} - 7308800z^{13} + 7630848z^{12} + 3035520z^{11} + 604800z^{10}$.

The second theorem

Two explicit formulas

For $n > 1$

$$B_1(n,2) = \frac{(n-2)(n-3)(13n^2-5n+4)}{3} \cdot 4^{n-6}.$$

For $n > 2$

$$B_2(n,2) = \frac{2(455n^4-1366n^3+1859n^2-650n+192)}{15} \cdot C_{n-3}^3 \cdot 4^{n-10}.$$